THE RELEVANCE OF PRIMES IN BRAUER–MANIN OBSTRUCTION

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This note attempts to summarize results on the relevance of primes in Brauer-Manin obstruction, with an emphasis on K3 surfaces.

1. Notations

(1) k is a global field. For a finite prime v: k_v is the completion, $\mathcal{O}_v$ its ring of integers, $\mathbb{F}_v$ its residue field with algebraic closure $\overline{\mathbb{F}}_v$, p the residue characteristic, and e_v is the absolute ramification index of v, that is the ramification index of $k_v/\mathbb{Q}_p$, equivalently of v in $k/\mathbb{Q}$.
(2) X/k is a smooth, projective and geometrically integral variety.
(3) $\mathrm{Br}_0(X)$ is the image of $\mathrm{Br}(k) \to \mathrm{Br}(X)$, and $\mathrm{Br}_1(X)$ is the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{k}})$.
(4) Fix a prime v. Then $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \mathrm{Br}(k_v)$ is evaluation of a class $\mathcal{A} \in \mathrm{Br}(X_{k_v})$.
(5) A prime v is *relevant* if some $\mathcal{A} \in \mathrm{Br}(X)$ has non-constant $\mathrm{ev}_{\mathcal{A}}$ on $X(k_v)$; otherwise it is *irrelevant*.
(6) A prime v is *potentially relevant* if there exists a finite field extension k'/k and a prime w of k' over v such that w is relevant.
(7) At a prime v of good reduction, fix $\mathcal{X}_v$ a smooth model over $\mathcal{O}_v$. Let $\mathcal{X}_{\mathbb{F}_v}$ be its special fibre.

2. Number fields, relevance over k

Let k be a number field. The goal is to find (if exists) a finite set S of primes, containing the archimedean primes, and a closed set $Z \subset \prod_{v\in S} X(k_v)$ with

$$X(\mathbb{A}_k)^{\mathrm{Br}} = Z \times \prod_{v\notin S} X(k_v).$$

In particular, if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, S contains all the relevant primes if and only if $X(\mathbb{A}_k)^{\mathrm{Br}}$ decomposes as above. In this case, $v \notin S$ does not play a role in the Brauer Manin obstruction to either Hasse principle or weak approximation. This leads to the study whether the evaluation map at a class $\mathcal{A} \in \mathrm{Br}(X)$ is constant.

A sufficient criterion for constant evaluation says: if a class $\mathcal{A} \in \mathrm{Br}(X_{k_v})$ is in the subgroup generated by the image of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$, $\mathrm{ev}_{\mathcal{A}}$ on $X(k_v)$ is constant (we only need $\mathcal{X}_v$ to be proper).

2.1. Results at a prime of good reduction.

Fix a prime v of good reduction. The sufficient criterion above is satisfied in the following cases:

(1) *Algebraic classes.* [CS13, Proposition 2.3]: if $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{NS}(X_{\overline{k}})$ is torsion-free, then every class in $\mathrm{Br}_1(X_{k_v})$ has constant evaluation map on $X(k_v)$.

(2) *Classes of order prime to p.* [CS13, Proposition 2.4]: let $\ell \neq p$ be prime. Suppose $H^1_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_v}, \mathbb{Z}/\ell) = 0$. Then every class in the ℓ-primary subgroup $\mathrm{Br}(X_{k_v})\{\ell\}$ has constant evaluation map on $X(k_v)$.

Then Colliot-Thélène and Skorobogatov prove the following:

(1) *General.* [CS13, Theorem 3.1]: suppose $H^1(X, \mathcal{O}_X) = 0$, $\mathrm{NS}(X_{\overline{k}})$ is torsion free, and $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite. Then there are finitely many relevant primes, which are among:
   (a) the archimedean primes;
   (b) the primes of bad reduction;
   (c) the primes of good reduction where p divides $|\mathrm{Br}(X)/\mathrm{Br}_1(X)|$.

(2) *K3 version.* The hypotheses hold for K3 surfaces: the finiteness of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is [SZ08].

3. The list of good primes

Some primes can be relevant despite good reduction. For example, in [CS13, Theorem 3.1], the good primes dividing $|\mathrm{Br}(X)/\mathrm{Br}_1(X)|$ are not excluded. Whether a good prime is relevant is further investigated by Bright, Newton, Ieronymou, and Pagano.

3.1. **Bright–Newton.**

(1) *General.* [BN23, Theorem A]: assume v is a prime of good reduction. A class has constant evaluation over every finite extension of k_v if and only if its Kato Swan conductor is 0 and its residue in $H^1(\mathcal{X}_{\mathbb{F}_v}, \mathbb{Q}/\mathbb{Z})$ comes from $H^1(\mathbb{F}_v, \mathbb{Q}/\mathbb{Z})$. Theorem D below, [Ier22, Proposition 8], and [Pag25, Theorems 1.3 and 1.4] build on Kato Swan conductor and the residue map.

(2) *General.* [BN23, Theorem D]: assume $\mathrm{Pic}\, X_{\overline{k}}$ is finite free (no assumption on $\mathrm{Br}(X)/\mathrm{Br}_1(X)$). Then there are only finitely many relevant primes, which are among:
   (a) the archimedean primes;
   (b) the primes of bad reduction;
   (c) the primes of good reduction with $e_v \geq p - 1$;
   (d) the primes of good reduction such that for every smooth model $\mathcal{X}_v$, $H^0(\mathcal{X}_{\mathbb{F}_v}, \Omega^1) \neq 0$.

(3) *K3 version.* The last set (d) of [BN23, Theorem D] is empty for K3 surfaces. Therefore, a prime of good reduction with $e_v < p - 1$ is irrelevant. Over $\mathbb{Q}$, every good prime other than 2 is irrelevant. The first example over $\mathbb{Q}$ where 2 is a relevant good prime is in [Pag22].

3.2. **Ieronymou.**

(1) *General.* [Ier22, Proposition 6]: assume v is a prime of good reduction. Suppose $H^1_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_v}, \mathbb{Z}/p) = 0$. Then a p-primary class has constant evaluation on $X(k_v)$ if it has Swan conductor 0. In particular, this holds if $H^0(\mathcal{X}_{\mathbb{F}_v}, \Omega^1) = 0$ and $e_v < p - 1$ (cf. [BN23, Theorem D(c), (d)]).

(2) *K3 version.* The assumptions $H^1_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_v}, \mathbb{Z}/p) = 0$ and $H^0(\mathcal{X}_{\mathbb{F}_v}, \Omega^1) = 0$ are satisfied by K3 surfaces. In particular, this recovers the K3 version of [BN23, Theorem D].

3.3. **Pagano.**

(1) *General.* [Pag25, Theorem 1.3]: assume v is a prime of ordinary good reduction such that $H^0(\mathcal{X}_{\mathbb{F}_v}, \Omega^1) = 0$, $H^1_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_v}, \mathbb{Z}/p) = 0$, and $(p-1) \nmid e_v$. Then v is irrelevant.
(2) *K3 version.*
   (a) For K3 surfaces, the two cohomological hypotheses in [Pag25, Theorem 1.3] hold automatically, so a prime of good ordinary reduction v is irrelevant if $(p-1) \nmid e_v$ [Pag25, Theorem 1.3].
   (b) Another result only for K3 surfaces is [Pag25, Theorem 1.4]: a prime v of good non-ordinary reduction is irrelevant if $e_v \leq p-1$ [Pag25, Theorem 1.4].
(3) *Examples.*
   (a) For the criterion in [Pag25, Theorem 1.3], [Pag25, Example 4.11] and [Pag25, Theorem 6.2] with $\alpha = 1$ show that v can be either relevant or irrelevant if $(p-1) \mid e_v$.
   (b) For the criterion in [Pag25, Theorem 1.4], [Pag25, Theorem 6.2] with $\alpha = \sqrt{2}$ shows that if $e_v = p$, v can be relevant.

4. The list of bad primes

[CS13] puts every bad prime into S, which can be refined.

4.1. **Bright.** Suppose X has a regular model $\mathcal{X}_v$ at v. Let $\mathcal{X}^\circ_{\mathbb{F}_v}$ be the smooth locus of the $\mathcal{X}_{\mathbb{F}_v}$. Let $n > 0$ be prime to p.

(1) [Bri15, Corollary 5.7]: if $H^1(\mathcal{X}^\circ_{\overline{\mathbb{F}}_v}, \mathbb{Z}/n) = 0$, every class in $\mathrm{Br}(X_{k_v})[n]$ has constant evaluation on $X(k_v)$.
(2) [Bri15, Proposition 5.9]: every class in $\mathrm{Br}(X_{k_v})[n]$ with trivial geometric residue has constant evaluation map on $X(k_v)$.
(3) [Bri15, Theorem 5.16]: if the geometric residue of a class in $\mathrm{Br}(X_{k_v})$ is non-zero and $|\mathbb{F}_v|$ is large enough, then it has non-constant evaluation on $X(k_v)$.

4.2. **Ieronymou.** Suppose X has a regular model $\mathcal{X}_v$ at v where the irreducible components of $\mathcal{X}_{\mathbb{F}_v}$ are geometrically integral.

(1) [Ier22, Proposition 19]: suppose $\mathcal{A} \in \mathrm{Br}(X_{k_v})$ becomes zero over the maximal unramified extension of k_v. Then for every $x \in X(k_v)$, $\mathrm{ev}_{\mathcal{A}}(x)$ depends only on the component x reduces to.
(2) If $H^1(I_v, \mathrm{Pic}\, X_{\overline{k}_v}) = 0$, where I_v is the inertia group of k_v, the same holds for all classes in $\mathrm{Br}_1(X_{k_v})$ [Ier22, Corollary 21].

5. Number fields, potential relevance

The list of relevant primes can change under base change. This leads to the notion of potential relevance and Wittenberg's question.

5.1. **Potential relevance.** We first give examples where the relevance of a prime persists/changes after field extensions.

In [CS13, Propositions 2.3 and 2.4], the hypotheses are geometric or valid over every field extension. Therefore at good primes, algebraic classes and classes of order prime to p stay irrelevant over every finite extension.

However, the third item of the list in [CS13, Theorem 3.1], the primes dividing the order of the transcendental Brauer group, is not stable under base change: indeed, the group $\mathrm{Br}(X_{\overline{k}})$ contains $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$, where b_2 is the second Betti number and ρ is the geometric Picard rank. If $b_2 > \rho$ (for example when $H^2(X, \mathcal{O}_X) \neq 0$), the transcendental Brauer group can get larger after base change.

5.2. **Wittenberg's questions.** Suppose X satisfies the Hasse principle and weak approximation over every finite extension. Wittenberg asked, in decreasing order of strength:

(1) must X be rationally connected;
(2) for a fixed i, must $H^i(X, \mathcal{O}_X) = 0$?

5.2.1. *When $i = 1$.* Wittenberg's question has a positive answer for $i = 1$ as a result of [Har00, Corollary 2.4], as pointed out in [BN23].

5.2.2. *When $i = 2$.*

(1) [BN23, Theorem C]: suppose $H^2(X, \mathcal{O}_X) \neq 0$. Then every prime of good ordinary reduction is potentially relevant. This gives a positive answer to Wittenberg's question partially for $i = 2$, assuming the existence of a good ordinary prime. (Such primes are conjectured to have positive density, and this is known for abelian surfaces and K3 surfaces.)
(2) [ANP25, Corollary 1.1.3]: suppose $H^2(X, \mathcal{O}_X) \neq 0$. Then almost all primes are potentially relevant. This gives a positive answer to Wittenberg's question for $i = 2$.

6. Global function fields

The sections above focus on the case when k is a number field. Now we focus on the case when k is a global function field of characteristic p.

6.1. **How the picture changes.**

6.1.1. *Unchanged.* The analogues of [CS13, Proposition 2.3 and Proposition 2.4] hold over function fields:

(1) The condition in [CS13, Proposition 2.3] is replaced by the torsion-freeness of $\underline{\mathrm{Pic}}_{X/k}$: they are equivalent in characteristic 0 and the latter is stronger than the former in positive characteristic. This is pointed out in the proof of [LS26, Proposition 4.1].
(2) Similarly, for $\ell \neq p$, the condition in [CS13, Proposition 2.4] can be replaced by $\underline{\mathrm{Pic}}_{X/k}[\ell] = 0$, as a result of [LS26, Proposition 2.1].

6.1.2. *Changed.* Every prime now has the same residue characteristic $p = \mathrm{char}\, k$. In particular, in the list of [CS13, Theorem 3.1], the third item gives either all primes or nothing, depending on whether p divides $|\mathrm{Br}(X)/\mathrm{Br}_1(X)|$.

Also, the finiteness of $|\mathrm{Br}(X)/\mathrm{Br}_1(X)|$ for K3 surfaces can fail over k in the supersingular case, in which case $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is killed by p by [LS24].

6.2. **The unipotent Brauer group.** The quotient of $\mathrm{Br}(X_{\overline{k}})[p^\infty]$ by its divisible subgroup is the group of $\bar{k}$-points of a quasi-algebraic group, an extension of a finite étale group D_X by a smooth connected unipotent group U_X, the *unipotent Brauer group* [LS26, introduction]. For K3 and abelian surfaces, $U_X \neq 0$ if and only if X is supersingular.

6.3. **The dichotomy.** [LS26, Theorem A]: Let ℓ be a prime.

(1) If $\ell = p$: when $U_X = 0$ and $\underline{\mathrm{Pic}}_{X/k}[p] = 0$, only finitely many primes are potentially relevant to the p-primary Brauer–Manin obstruction; otherwise almost all primes are.
(2) If $\ell \neq p$: when $\underline{\mathrm{Pic}}_{X/k}[\ell] = 0$, only finitely many primes are potentially relevant to the ℓ-primary obstruction; otherwise almost all are.

In particular, we have [LS26, Corollary B]: if $U_X = 0$ and $\underline{\mathrm{Pic}}_{X/k}$ is torsion-free, only finitely many primes are potentially relevant to the Brauer–Manin obstruction; otherwise almost all are. This extends the result of [BN23, Theorem D].

6.4. **Constant varieties.** Let $\mathbb{F}$ be the constant subfield of k, i.e. the largest algebraic sub-extension of $k/\mathbb{F}_p$. Suppose X descends to $\mathbb{F}$. Then the dichotomy simplifies into [LS26, Theorem C]:

(1) If $U_X = 0$ and $\underline{\mathrm{Pic}}_{X/\mathbb{F}}$ is torsion-free, then $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)$.
(2) If $U_X \neq 0$, or $\underline{\mathrm{Pic}}_{X/\mathbb{F}}[\ell] \neq 0$ for some prime ℓ, then every prime of k is potentially relevant.

6.5. **p-torsion free geometric Brauer group.**

(1) *General.* [LS26, Proposition 4.1]: if $\underline{\mathrm{Pic}}_{X/k}$ is torsion-free and $\mathrm{Br}(X_{\overline{k}})[p^\infty] = 0$, then no prime of good reduction is potentially relevant.
(2) *Surfaces.* [LS26, Proposition 4.2]: for a surface with $\underline{\mathrm{Pic}}_{X/k}$ smooth and $U_X = 0$: if $b_2 = \rho + 2h$ and $\mathrm{NS}(X_{\overline{k}})[p^\infty] = 0$, then $\mathrm{Br}(X_{\overline{k}})[p^\infty] = 0$. Here ρ is the geometric Picard rank, b_2 the second Betti number, and $h = \dim_K H^2(X_{\overline{k}}, W\mathcal{O}_{X_{\overline{k}}}) \otimes K$ with W the Witt ring of $\bar{k}$ and K its fraction field.
(3) *K3 version.* For a K3 surface, the Picard scheme is torsion-free, and

$$U_X = 0 \iff h > 0 \iff X \text{ is non-supersingular}$$

Then the hypothesis of [LS26, Proposition 4.1] amounts to $\rho + 2h = 22$ with $h > 0$ by [LS26, Proposition 4.2]. Examples are [LS26, Example 4.3 and Proposition 4.4].

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